Maximum Flow is Fair: A Network Flow Approach to Committee Voting

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In the committee voting setting, a subset of k alternatives is selected based on the preferences of voters. In this paper, our goal is to efficiently compute *ex-ante* fair probability distributions (or lotteries) over committees. Since it is not known whether a lottery satisfying the desirable fairness property of *fractional core* is polynomial-time computable, we introduce a new axiom called *group resource proportionality* (GRP), which strengthens other fairness notions in the literature. We characterize our fairness axiom by a correspondence with max flows on a network formulation of committee voting. Using the connection to flow networks revealed by this characterization, we then introduce voting rules which achieve fairness in conjunction with other desirable properties. The *redistributive utilitarian rule* satisfies ex-ante efficiency in addition to our fairness axiom. We also give a voting rule which maximizes social welfare subject to fairness by reducing to a minimum-cost maximum-flow problem. Lastly, we show our fairness property can be obtained in tandem with strong *ex-post* fairness properties – an approach known as *best-of-both-worlds* fairness. We strengthen existing best-or-both-worlds fairness results in committee voting and resolve an open question posed by Aziz et al. [6]. These findings follow from an auxiliary result which may prove useful in obtaining best-of-both-worlds type results in future research on committee voting.

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1 INTRODUCTION

In the *committee voting* problem, we are tasked with selecting a subset of k alternatives (or *candidates*) given the preferences of a population of voters. The problem models a rather natural and practical scenario and has garnered notable attention in recent research [2, 11, 22]. In this work, we contribute to the growing body of work pursuing fair committee voting outcomes when voters report preferences in the form of a subset of candidates of which they *approve* (see Lackner and Skowron [33] for extensive coverage of this topic). One clear advantage of approval ballots is due to their simplicity compared to ballots such as rankings and cardinal reports. Furthermore, approval ballots are clearly the most natural method of preference elicitation under our assumptions about voters' preference structure: we assume voters' preferences are *dichotomous* with respect to candidates and derive utility equal to the number of approved candidates on the selected committee.

While the assumption of dichotomous preferences allows for many theoretical results that encounter impossibilities in more general settings, there remain significant obstacles in obtaining committees which are fair to all voters. Most fundamentally, no deterministic rule satisfies a very basic fairness notion known as *positive share*, which guarantees that every voter receives some non-zero representation on the selected committee. A natural approach to circumvent this and other impossibilities is to employ randomization by computing a probability distribution (or *lottery*) over committees which achieves desirable properties in an *ex-ante* sense. This approach has significant precedent in the (single-winner) voting setting [29], and in particular, voting under dichotomous preferences [1, 9, 10].

Recently, this approach has been extended to the committee voting setting with dichotomous preferences [6, 18]. Fairness axioms in committee voting often aim for some semblance of proportional representation, the idea that every group of voters should control a fraction of the committee proportional to their size. The most pervasive such axiom is that of *core*, initially conceptualized and investigated by Droop [19] and Lindahl [35]. While existence of the core is not known in an ex-post sense, its fractional analog known as *fractional core* has been shown to exist. However, there is no known polynomial time algorithm for computing fractional core [37]. Furthermore, while maximizing Nash welfare computes a fractional core outcome in the single-winner setting, this does not extend to committee voting (see Example 4.1), highlighting the structural differences between the two settings. Indeed, Aziz et al. [6] demonstrate the technical subtleties that arise when extending axioms and algorithms from probabilistic single-winner voting to probabilistic committee voting. They extend the fair share axioms from the single winner setting [9, 20] to the committee setting, showing that two alternative interpretations of those axioms result in distinct hierarchies, the strongest properties of which are *group fair share* and *strong unanimous fair share*.

Our first contribution is the *group resource proportionality* (GRP) axiom (Section 3), which unifies both axiom hierarchies from Aziz et al. [6] (by strengthening both group fair share and strong unanimous fair share), and is achievable in polynomial time. At a high level, GRP lower bounds, for each group of voters S, the number of selected candidates which represent some voter in S, in expectation. The lower bound depends on the proportional size and the preference structure of the voter group. We provide a characterization of GRP (Theorem 3.3) using a network flow formulation of our problem, in which voters control a proportional fraction of the committee size (or "budget"), and can only flow into candidates they approve. Specifically, we show that a lottery satisfies our axiom if and only if it corresponds to a solution to the max flow problem on this network. We point out that, since we show GRP is implied by fractional core (Proposition 3.8), this characterization also sheds some light on the structure of fractional core solutions, about which not much is known.

In Section 4, we exploit the connection drawn between probabilistic committee voting and network flows in order to devise fair voting rules which achieve additional desiderata. We introduce a voting rule which obtains GRP and ex-ante (Pareto) efficiency and can be computed in polynomial time. Our algorithm – the *redistributive utilitarian rule* – leverages our characterization and carefully redistributes voters' remaining budgets to avoid any voter exhausting their budget prematurely. In the single-winner setting, Brandl et al. [10] showed that ex-ante efficiency and strategyproofness cannot be achieved jointly with positive share. In that setting, the conditional utilitarian rule (CUT) offers a compromise: strategyproofness and efficiency subject to group fair share. We give an impossibility demonstrating that no rule satisfying these properties exists in probabilistic committee voting. We introduce *generalized CUT*, a voting rule which maximizes social welfare subject to GRP. To do so, we frame instances of our problem using a minimum-cost maximum-flow formulation, and obtain a rule which is strategyproof and utilitarian welfare optimal subject to GRP (and thus efficient subject to GRP).

In Section 5, we show ex-ante GRP can be obtained in tandem with strong ex-post fairness properties, such as FJR and EJR+, contributing to the literature on "best of both worlds fairness" [6, 7]. To do so, we identify a condition on an integral committee W which is sufficient to guarantee polynomial computation of an ex-ante GRP lottery over committees which contain W (Theorem 5.3). As corollaries of this statement, we get the strongest known "best of both worlds"-type results in the committee voting setting, and resolve an open question posed by Aziz et al. [6].

Related Work

We remark that many of the related work have already been discussed in the introduction. Here, we will discuss additional related works.

Approval-based committee voting is a fundamental voting model with broad applications. Multi-winner voting is readily applicable in settings where a fixed committee size must be chosen, such as in parliamentary elections , and the selection of a board members. Due to its generality, in recent years, it has found applications in recommender systems [16, 28, 42], the design of Q & A platforms [32], blockchain protocols [14], and global optimization [24].

As the existence of core is a major open problem in approval-based committee voting [34], there has been a significant body of work focused on developing algorithms that offer approximate notions of core. Peters and Skowron [40] showed that the Proportional Approval Voting rule gives a committee that is a factor two multiplicative approximation to core. Munagala et al. [37] show that a constant factor approximation to core is achievable in committee voting, even when the utility functions of the voters are monotone submodular. Another way to study relaxation of core is to restrict the deviating groups. This was the approach taken by Aziz et al. [3], who initiated the study of representation for cohesive groups, spawning a large body of work exploring various axioms and algorithms [4, 12, 13, 21].

Our paper achieves fairness in committee voting through a probabilistic viewpoint. The study of probability distributions over outcomes for a fixed profile of preferences is referred to as *probabilistic voting*. In single-winner probabilistic voting, Bogomolnaia et al. [9] defined various ex-ante fairness notions and algorithms, which has been further explored in later works [1, 10, 20, 36]. In the committee voting setting, Munagala et al. [37] used the existence of Lindahl equilibria along with various rounding schemes to prove existence of constant factor approximate core committees. Cheng et al. [18] introduced a concept known as *stable lottery*, which asserts that for any committee W of size α, the expected number of voters who prefer W to the committee sampled from the lottery does not exceed $\alpha\frac{n}{k}$ for each $1 \leq \alpha \leq k$. They use the probabilistic method to show the existence of stable lotteries. Unlike our fairness notion, no efficient algorithm is known to give stable lotteries under dichotomous preferences.

When studying probability distributions over outcomes that are fair in expectation, it is natural to ask whether such a distribution can be supported over desirable outcomes. This is referred to as "best-of-both-worlds fairness," a term coined by Freeman et al. [26]. They show that ex-ante envy-freeness and ex-post near envy-freeness is achievable in the resource allocation setting. Consequent works [25, 25, 31] have established similar guarantees are possible in various resource allocation problems. In approval-based committee voting, Aziz et al. [6] show that ex-ante GFS and ex-post EJR can be achieved simultaneously. This type of public goods best-of-both-worlds perspective was further explored by Aziz et al. [7] in the setting of participatory budgeting.

Our approach to exploring ex-ante fairness makes use of network flows, where flow conservation naturally aids in the feasible exchange of probability weights among voters. A similar flow-based approach was used by Vazirani [44] for the computation of market equilibria in a private goods economy and when studying competitive equilibria in trade networks [15].

2 PRELIMINARIES

For any positive integer $t \in \mathbb{N}$, we write $[t] := \{1, 2, \dots, t\}$. Denote C as the set of m *candidates* and $N = [n]$ as the set of voters. We assume that the voters have *dichotomous* preferences over candidates. That is, each voter's preference is represented by a set of approved candidates, called an *approval set*. For each voter $i \in N$, denote $A_i \subseteq C$ as the approval set of voter i. An instance $\mathcal{I}$ of approval-based committee voting is given by a set of candidates C, an *approval profile* $\mathcal{A} = (A_1, A_2, \dots, A_n)$, and a positive integer k with $k \leq m$.

Integral and Fractional Committees. An *integral committee* W (or simply committee, when context is clear) is a subset of C of size k. A *fractional committee* is specified by an m-dimensional vector $\vec{p} = (p_c)_{c \in C} \in [0, 1]^m$ with $\sum_{c \in C} p_c = k$. For notational convenience, we use $\vec{1}_W \in \{0, 1\}^m$ to denote the vector representation of an integral committee W, i.e. the fractional committee with the j^{th} component equal to 1 if and only if $j \in W$.

Lotteries. A *lottery* is a probability distribution over integral committees. Lottery is specified by a set of $s \in \mathbb{N}$ tuples $\{(\lambda_j, W_j)\}_{j \in [s]}$ with $\sum_j \lambda_j = 1$, where for each $j \in [s]$, the integral committee $W_j \subseteq C$ is selected with probability $\lambda_j \in [0, 1]$. It can be seen that every lottery $\{(\lambda_j, W_j)\}_{j \in [s]}$ has a unique corresponding fractional committee given by $\vec{p} = \sum_{j \in [s]} \lambda_j \vec{1}_{W_j}$. Here p_c can be interpreted as the marginal probability of candidate c being selected in a committee drawn from the lottery. We say lottery $\{(\lambda_j, W_j)\}_{j \in [s]}$ is an *implementation* of a fractional committee $\vec{p}$ if $\vec{p} = \sum_{j \in [s]} \lambda_j \vec{1}_{W_j}$.

As is common in approval-based committee voting, we assume voters' preferences over committees are determined by the number of approved candidates on the committee, i.e. $u_i(W) = |A_i \cap W|$. Extending to fractional committees, we define the utility of voter i from fractional committee $\vec{p}$ as $u_i(\vec{p}) := \sum_{c \in A_i} p_c$. We point out that voter i derives utility from a fractional committee $\vec{p}$ equivalent to his expected utility from any lottery implementing $\vec{p}$, i.e. $u_i(\vec{p}) = \mathbb{E}_{W \sim \Delta}[u_i(W)]$for any implementation Δ of $\vec{p}$.

A *(voting) rule* F is a function that maps each approval profile $\mathcal{A}$ and committee size k to a feasible fractional committee $F(\mathcal{A}, k) \in [0, 1]^m$.

Definition 2.1 (Strategyproofness)**.** A rule F is *strategyproof* if for all $i \in N$, all $k \leq m$, and all approval profiles of the form $\mathcal{A} = (A'_1, \dots, A_i, \dots, A'_n)$ and $\mathcal{A}' = (A'_1, \dots, A'_i, \dots, A'_n)$, we have $u_i(F(\mathcal{A}, k)) \geq u_i(F(\mathcal{A}', k))$.

Definition 2.2 (Efficiency)**.** A fractional committee $\vec{p}$ is *efficient* if there is no alternative fractional committee $\vec{q}$ such that $u_i(\vec{q}) \geq u_i(\vec{p})$ for all $i \in N$ and this inequality is strict for at least one agent.

We point out that, while we define voting rules and most of the properties in this paper in terms of fractional committees, each of our properties and results have natural analogs for lotteries. Specifically, we say a lottery satisfies a property ex-ante if and only if the lottery implements a

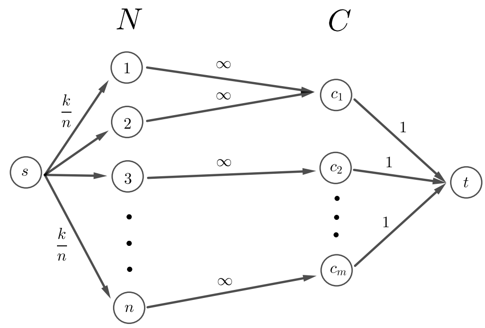

Fig. 1. Illustration of the network representation of an instance of committee voting.

fractional committee which satisfies the fractional property. As an example, if fractional committee $\vec{p}$ satisfies strategyproofness (as above), then any algorithm that outputs an integral committee sampled from a lottery which implements $\vec{p}$ will in fact satisfy *ex-ante* strategyproofness. Although every lottery corresponds to a unique fractional committee, the converse is not true. However, computation of an implementation of a fractional committee can be achieved through rounding schemes commonly studied in combinatorial optimization [17, 27, 43]. In particular, the rounding scheme of [5] shows that every fractional committee of size k can be implemented with a lottery over integral committees of size k in polynomial time. This means that the algorithmic results of Section 4 also carry over to lotteries. In summary, while we will often restrict our treatment to fractional committees, this is purely for the sake of notational clarity.

Flow Network Formulation

We will often represent committee voting instances using flow networks. At a high level, the network representation of an instance (i) connects the source to a node for each voter, (ii) connects each voter to candidates of which they approve, and (iii) connects those candidates to the sink. Refer to Figure 1 for an illustration of a network representation of an instance.

Definition 2.3. The *network representation* of an instance $\mathcal{I}$, denoted $\mathcal{N}_{\mathcal{I}}$ (or simply $\mathcal{N}$), is a flow network with source s, sink t, a node for each voter and each candidate, and edge capacities defined as follows (an edge exists if and only if a capacity is defined below):

- $cap(s,i) = k/n \quad \forall i \in N$
- $cap(i,c) = \infty \quad \forall i \in N, c \in A_i$
- $cap(c,t) = 1 \quad \forall c \in C$

Since we typically consider a single instance at a time, we will almost always use $\mathcal{N}$ to denote $\mathcal{N}_{\mathcal{I}}$. For $T \subseteq C$, we will define $\mathcal{N}(T)$ to be the network $\mathcal{N}_{\mathcal{I}}$ where vertices corresponding to candidates in $C \setminus T$ and associated arcs are removed. Formally, if $\mathcal{I} = (C, \mathcal{A}, k)$, then $\mathcal{N}(T) = \mathcal{N}_{I'}$ where $I' = (T, \mathcal{A}', k)$ and $\mathcal{A}' = (A_1 \cap T, \ldots, A_n \cap T)$. In this paper, we will often search for fair committees using the network flow representation of our instance. To do so, we must also be able to translate from network flows to fractional committees. Given a feasible flow f on the network representation of an instance, we define the *fractional committee given by* f to be $\vec{p}$ where $p_c = f(c \to t)$ for all $c \in C$. We remark that any fractional committee given by a feasible flow must also be feasible. This holds since $p_c = f(c \to t) \leq cap(c,t) = 1$ for all $c \in C$, and $\sum_{c \in C} f(c \to t) = \sum_{i \in N} f(s \to i) \leq k$.

3 A CHARACTERIZATION OF FAIR COMMITTEES

In this section, we introduce a new axiom which unifies and strengthens existing notions of fairness. We then give a characterization of our fairness notion using the network representation of the problem. We first give an intuition and motivation behind our fairness notion.

Definition 3.1 (Fractional Core)**.** A fractional committee $\vec{p}$ satisfies *fractional core* (or lies in the core) if there is no group of voters S and committee $\vec{q}$ with $\sum_{c\in C} q_c \leq |S|\frac{k}{n}$ such that $u_i(\vec{q}) \geq u_i(\vec{p})$ for all $i \in S$ with at least one inequality strict.

The fractional core has a *fair taxation* interpretation [35, 37]. The quantity $\frac{k}{n}$ can be thought of as the tax contribution of a voter. The tax money collected is used to buy candidates, where each candidate has a unit cost.[1] A committee satisfies the fractional core if no group of voters S collectively could use their tax contribution $|S|\frac{k}{n}$ to "buy" a fractional committee which they all prefer. Fractional core captures the notion that no group of voters, using only their own resources, can achieve a better outcome for themselves. By comparison, our fairness notion states that the total amount of tax contribution (resources) spent on candidates approved by S must be at least the maximum amount that the group S could spend, provided that no agent's contribution is spent on a candidate they do not approve. Thus, our fairness notion captures the spirit of fractional core while being easier to achieve.

Definition 3.2 (Group Resource Proportionality)**.** A fractional committee $\vec{p}$ is said to satisfy *group resource proportionality* (GRP) if for every $S \subseteq N$:

$$\sum_{c\in \bigcup_{i\in S} A_i} p_c \geq |S|\frac{k}{n} - \max_{T\subseteq S}\left[|T|\frac{k}{n} - |\bigcup_{i\in T} A_i|\right]. \tag{1}$$

It's important to note that the right hand side of inequality (1) is not necessarily $|S|\frac{k}{n}$, which one might expect, considering the fair taxation interpretation in which each voter in S contributes $\frac{k}{n}$. Instead, due to the fact that candidates can receive at most one unit of resources, this quantity depends on the structure of the approval sets of the voters in S.

Our fairness notion has a natural interpretation in terms of lotteries. Any lottery Δ implementing a GRP fractional committee $\vec{p}$ satisfies, for every group of voters $S \subseteq N$,

$$\mathbb{E}_{W\sim\Delta}[|\{c \in W \mid \exists i \in S \text{ such that } c \in A_i\}|] \geq |S|\frac{k}{n} - \max_{T\subseteq S}\left[|T|\frac{k}{n} - |\bigcup_{i\in T} A_i|\right],$$

where the left hand side is the expected number of selected candidates approved by some voter in S. Thus, lotteries satisfying GRP give a strong ex-ante representation guarantee to every coalition of voters.

GRP Characterization

As GRP gives strong representation guarantees for every coalition of voters, it is not clear a priori whether such fractional committees always exist or how to obtain one. In the following result, we use the network flow formulation of our problem to provide a characterization of outcomes that satisfy GRP.

Theorem 3.3. *A fractional committee $\vec{p}$ satisfies group resource proportionality if and only if there exists a max flow f on the network representation $\mathcal{N}$ such that $p_i \geq f(c_i \to t)$ for each $c_i \in C$.*

[1] Candidates can be bought fractionally, meaning that if in total $\alpha \in (0, 1)$ dollars are spent on a candidate c, then the candidate is selected fractionally with $p_c = \alpha$.

Proof. Let $\vec{p}$ be a fractional committee satisfying GRP, and consider a modified network $\widetilde{\mathcal{N}}$ with modified capacities $cap(c_i \to t) = p_i$ for each $c_i \in C$. By the max-flow min-cut theorem, there exists a maximum (or max) flow f on $\widetilde{\mathcal{N}}$ such that

$$\begin{aligned}
\sum_{c \in C} f(c \to t) &= \min_{T \subseteq N} \left[|N \setminus T| \frac{k}{n} + \sum_{c \in \bigcup_{i \in T} A_i} p_c \right] \\
&= |N \setminus T^*| \frac{k}{n} + \sum_{c \in \bigcup_{i \in T^*} A_i} p_c \\
&\geq |N \setminus T^*| \frac{k}{n} + |T^*| \frac{k}{n} - \max_{T \subseteq T^*} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right] \\
&= k - \max_{T \subseteq T^*} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right] \\
&\geq k - \max_{T \subseteq N} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right]
\end{aligned}$$

where the first inequality follows from the fact that $\vec{p}$ satisfies GRP. Observe that

$$k - \max_{T \subseteq N} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right] = \min_{T \subseteq N} \left[|N \setminus T| \frac{k}{n} + |\bigcup_{i \in T} A_i| \right]$$

which is equal to the min cut value of $\mathcal{N}$. Hence, f is indeed a max flow on $\mathcal{N}$ and since f is also a valid flow on $\widetilde{\mathcal{N}}$, it satisfies $p_i \geq f(c_i \to t)$ for each $c_i \in C$.

We now prove the converse direction. Let f be a max flow on network formulation $\mathcal{N}$ and $p_i \geq f(c_i \to t)$ for each $c_i \in C$. Suppose, on the contrary, that $\vec{p}$ does not satisfy GRP. That is, there exists $S' \subseteq N$ such that

$$\sum_{c \in \bigcup_{i \in S'} A_i} p_c < |S'| \frac{k}{n} - \max_{T \subseteq S'} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right]. \tag{2}$$

Observe that $|S'| \frac{k}{n} - \max_{T \subseteq S'} \left[|T| \frac{k}{n} - |\bigcup_{i \in T} A_i| \right] = \min_{T \subseteq S'} \left[|S' \setminus T| \frac{k}{n} + |\bigcup_{i \in T} A_i| \right]$, which is the min cut value of a subnetwork $\mathcal{N}'$ where the set of voters is S' and the set of candidates is $\bigcup_{i \in S'} A_i$. By the max-flow min-cut theorem, we know that there exists a max flow f' on $\mathcal{N}'$ whose value satisfies $\sum_{c \in \bigcup_{i \in S'} A_i} f'(c \to t) = \min_{T \subseteq S'} \left[|S' \setminus T| \frac{k}{n} + |\bigcup_{i \in T} A_i| \right]$. Consider a new flow f^* on the entire network $\mathcal{N}$ defined as follows:

$$f^*(s \to i) = f'(s \to i) \;\; \forall i \in S', \qquad f^*(i \to c_j) = f'(i \to c_j) \;\; \forall i \in S', \forall c_j \in A_i$$

$$f^*(c_j \to t) = f'(c_j \to t) \;\; \forall c_j \in \bigcup_{r \in S'} A_r$$

$$f^*(s \to i) = f(s \to i) \;\; \forall i \in N \setminus S' \qquad f^*(i \to c_j) = 0 \;\; \forall i \in N \setminus S',\; \forall c_j \in \bigcup_{i \in S'} A_i$$

$$f^*(i \to c_j) = f(i \to c_j) \;\; \forall i \in N \setminus S',\; \forall c_j \in A_i \setminus \bigcup_{r \in S'} A_r \qquad f^*(c_j \to t) = f(c_j \to t) \;\; \forall c_j \in C \setminus \bigcup_{r \in S'} A_r$$

Note that f^* satisfies the capacity constraints as well as the flow conservation constraints on the entire network $\mathcal{N}$, and thus is a valid flow on $\mathcal{N}$. Finally we see that,

$$\begin{aligned}
\sum_{c\in C} f(c\to t) &= \sum_{c\in \bigcup_{i\in S'} A_i} f(c\to t) + \sum_{c\in C\setminus \bigcup_{i\in S'} A_i} f(c\to t) \\
&\leq \sum_{c\in \bigcup_{i\in S'} A_i} p_c + \sum_{c\in C\setminus \bigcup_{i\in S'} A_i} f(c\to t) \qquad \because f(c_i)\leq p_i \ \forall c_i\in C \\
&< \left[|S'|\frac{k}{n} - \max_{T\subseteq S'}\left[|T|\frac{k}{n} - |\bigcup_{i\in T} A_i|\right]\right] + \sum_{c\in C\setminus \bigcup_{i\in S'} A_i} f(c\to t) \qquad \because by \ (2) \\
&= \min_{T\subseteq S'}\left[|S'\setminus T|\frac{k}{n} + |\bigcup_{i\in T} A_i|\right] + \sum_{c\in C\setminus \bigcup_{i\in S'} A_i} f(c\to t) \\
&= \sum_{c\in \bigcup_{i\in S'} A_i} f'(c\to t) + \sum_{c\in C\setminus \bigcup_{i\in S'} A_i} f(c\to t) \\
&= \sum_{c\in C} f^*(c\to t)
\end{aligned}$$

However, this contradicts the assumption that f is a max flow on $\mathcal{N}$. □

An immediate consequence of Theorem 3.3 is that every fractional committee given by a max flow f on the network representation $\mathcal{N}$ satisfies GRP. As we will see, the characterization result also provides a useful tool for designing rules which satisfy GRP.

Comparison With Existing Fairness Notions

In this section, we compare our fairness notion with the existing fairness notions. We begin by noting that, due to the structural differences between the two settings, significant challenges arise when extending axioms and algorithms from the single-winner setting of probabilistic voting to probabilistic committee voting. This was explored by Aziz et al. [6], who extended the fair share axioms from the single-winner setting [9, 20] to the committee voting setting, showing that two alternative interpretations are possible resulting in two distinct fairness hierarchies. The strongest axioms of the two distinct hierarchies are *group fair share* (GFS) and *strong unanimous fair share* (Strong UFS).

Definition 3.4 (Strong UFS [6])**.** A fractional committee $\vec{p}$ satisfies *Strong UFS* if for all $S\subseteq N$ where $A_i = A_j$ for any $i, j\in S$, it holds for each $i\in S$ that $u_i(\vec{p}) = \sum_{c\in A_i} p_c \geq \min\left\{|S|\frac{k}{n}, |A_i|\right\}$.

Definition 3.5 (GFS [6])**.** A fractional committee $\vec{p}$ satisfies *GFS* if it holds for every $S\subseteq N$ that $\sum_{c\in\bigcup_{i\in S} A_i} p_c \geq \frac{1}{n}\cdot\sum_{i\in S}\min\{k, |A_i|\}$.

We first show that our fairness axiom unifies both fairness hierarchies of [6] by strengthening both GFS and Strong UFS. All omitted proofs can be found in the appendix.

Proposition 3.6. *Group resource proportionality implies Strong UFS and GFS.*

Furthermore, we point out that if an integral committee satisfies GRP, then it also satisfies the well-studied proportional representation notion of PJR [see, e.g., 4, 34, 41]. An analog result does not hold for Strong UFS or GFS [6]. Since PJR is not a focus of this work, we defer its definition and the proof of the following proposition to the appendices.

Proposition 3.7. *If an integral committee satisfies GRP, then it satisfies PJR.*

We next show that fractional core implies GRP. Note this implies that for every committee $\vec{p}$ in the fractional core, there exists a max flow f on the network representation $\mathcal{N}$ such that $p_i \geq f(c_i \to t)$ for each $c_i \in C$ by Theorem 3.3. This provides additional insight into the structure of fractional core solutions, which is currently not well understood.

Proposition 3.8. *Fractional core implies group resource proportionality.*

Although GRP is implied by the fractional core, there are several advantages to our fairness notion. First, while no known efficient algorithm exists for the fractional core [37], our fairness notion can be achieved through an efficient algorithm. Second, given a fractional committee $\vec{p}$, checking whether it satisfies GRP can be done in polynomial time (Proposition 3.9). This is in contrast to the fractional core, where to the best of our knowledge, it is not known whether such an algorithm exists.

Proposition 3.9. *For a given fractional committee $\vec{p}$, checking whether $\vec{p}$ satisfies group resource proportionality can be done in polynomial time.*

4 ALGORITHMS FOR FAIR AND EFFICIENT COMMITTEES

Given there may be many max flow solutions to the network formulation of a given instance, and thus many solutions satisfying our fairness notion, we seek to further refine our solution set by searching for outcomes which are both fair and efficient.

In the single-winner setting, the voting rule which maximizes Nash welfare is ex-ante efficient and satisfies fractional core. However, as we will now show, the maximum Nash welfare rule does not guarantee GRP in the probabilistic committee voting setting, let alone fractional core.

Example 4.1 (Nash violates GRP). Suppose there are four voters, three candidates, $k = 2$, and the voters' preferences are: $A_1 = \{a\}, A_2 = A_3 = \{a, b\}$, and $A_4 = \{c\}$. Since $N_a \supsetneq N_b$, a dominates b with respect to Nash welfare and thus b can be selected with positive probability in the Nash welfare-maximizing outcome if and only if a is integrally selected. Thus, due to the committee size and number of candidates, it is clear that a is integrally selected. One can then frame maximum Nash welfare as a single-variable optimization to show that $\vec{p} = (1, \frac{1}{3}, \frac{2}{3})$ is the unique fractional committee which maximizes Nash welfare. However, denoting $S = \{1, 2, 3\}$, we can then see that $\vec{p}$ violates GRP with respect to S:

$$|S|\frac{k}{n} - \max_{T \subseteq S}\left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right] = \frac{3}{2} > \frac{4}{3} = \sum_{j \in \cup_{i \in S} A_i} p_j$$

where the first equality follows since the maximum is attained with the empty set.

It is worth pointing out that the incompatibility proved by Example 4.1 does not appear to result from a clash between efficiency and fairness. Indeed, the natural GRP fractional committee $\vec{q} = (1, \frac{1}{2}, \frac{1}{2})$ is ex-ante efficient. Furthermore, the fractional committee selected by the Nash rule in Example 4.1 is more egalitarian than any solution satisfying GRP.[2] Thus, Nash welfare embodies a sense of fairness distinct from that of fractional core and similar concepts, which measure fairness by comparing against a "deserved" outside option, as opposed to taking a welfarist approach. While these fairness notions coincide in the single-winner setting, they diverge in the committee voting setting. One possible intuition for this divergence is that voters' optimal utilities can be any integer in $[k]$ in the committee voting setting, whereas they are all equal to one in the single-winner

[2]Note that any fractional committee $\vec{q}$ with $q_c > \frac{1}{2}$ cannot elementwise dominate a max flow on $\mathcal{N}$.

ALGORITHM 1: Redistributive Utilitarian Rule (RUT)

Input: Voters N, candidates C, approval profile $(A_i)_{i\in N}$ and committee size k.
Output: A fractional committee $\vec{p} = (\Delta_c)_{c\in C}$ of size k.
$\lambda_i \leftarrow 1$ for all $i \in N$
$s^* \leftarrow \max_{c\in C} s_\lambda(c)$
Let f_0 denote a trivial flow on any flow network.
$j \leftarrow 1$
while $j \leq m$ **do**
  $c_j \leftarrow \arg\max_{c\in C} s_\lambda(c)$
  $\mathcal{N}^j \leftarrow \mathcal{N}(\{c_1, \ldots, c_j\})$
  Apply Lemma 4.2 to the flow f_{j-1} on $\mathcal{N}^j$ to obtain a max flow f_j on $\mathcal{N}^j$.
  $V_j \leftarrow \{i \in N : f_j(s \to i) < \frac{k}{n}\}$
  if $A_i \subseteq \{c \in C : f_j(c \to t) = 1\}\ \forall i \in V_j$ **then**
    Exit loop.
  end
  $\alpha \leftarrow \min \left\{ \alpha \in \mathbb{R} : \max_{c\in C\setminus\{c_1,\ldots,c_j\}} \alpha \cdot |N_c \cap V_j| + s_\lambda(c) = s^* \right\}$
  for $i \in V_j$ **do**
    $\lambda_i \leftarrow \lambda_i + \alpha$
  end
  $j \leftarrow j + 1$
end
$f^* \leftarrow f_j$
Denote $\vec{p}$ as the fractional committee given by f^*.
while $k - \sum_{c\in C} p_c > 0$ **do**
  $\delta \leftarrow k - \sum_{c\in C} p_c$
  $c \leftarrow \arg\max_{\{c\in C: p_c<1\}} s_\lambda(c)$
  $p_c \leftarrow \max(1, p_c + \delta)$
end
return $\vec{p}$

setting. This can increase heterogeneity in voters' "deserved" outside options, but the Nash rule maximizes a welfare function which depends on voters' utilities in a symmetric fashion.

Since we already know how to compute a fractional committee satisfying GRP in polynomial time, it is natural to ask whether we can achieve fairness and efficiency by iteratively identifying and applying Pareto improvements to a fair committee, which can be done with a simple linear program. However, even in the single-winner special case, an arbitrary Pareto improvement with respect to a GRP fractional committee need not maintain the weaker property of GFS, let alone GRP.[3] Thus, an approach which treats fairness and efficiency sequentially is likely to meet with significant obstacles. In the next subsection, we will present an algorithm that instead maintains the invariant of efficiency while constructing a fair fractional committee.

4.1 Redistributive Utilitarian Rule: Efficiency + GRP

At a high level, our algorithm – which we call the *redistributive utilitarian rule* (RUT) – maintains the invariant of efficiency while constructing a fair fractional committee. To do so, it ensures that the selected committee maximizes weighted utilitarian welfare while iteratively computing max flows and redistributing flow to avoid long augmenting paths. We will now give a more detailed description of RUT. Refer to Algorithm 1 for a detailed description in pseudocode.

We start with unit weights $\lambda_i = 1$ and a modified version of $\mathcal{N}$ such that all candidates are removed from the network. We identify some candidate c^* which maximizes over all $c \in C$ the "score" $s_\lambda(c) = \sum_{i \in N_c} \lambda_i$ and let $s^* = \sum_{i \in N_{c^*}} \lambda_i$. In each round, we add c^* to our modified network and compute a max flow f which balances flows in a way that avoids saturating any edge from the source to an agent whenever possible (see Lemma 4.2). The weights λ_i are then effectively frozen for any agents with $f(s \to i) = \frac{k}{n}$, and the weights of the other agents are uniformly increased until there is some new candidate c^* with $s_\lambda(c^*) = s^*$.

This loop terminates when, for each $i \in N$ with $f(s \to i) < \frac{k}{n}$, it holds that $A_i \subseteq \{c \in C : f(c \to t) = 1\}$. Intuitively, this means there is no agent with unsaturated capacity from the source who approves of some candidate with unsaturated capacity to the sink. We refer to the network flow resulting from this process as f^*. Lastly, letting $\vec{q}$ denote the fractional committee given by f^*, we greedily allocate the remaining probability $k - \sum_{c \in C} q_c$ to candidates in descending order of $s_\lambda(c)$, and return the resulting fractional committee, denoted $\vec{p}$.

A subtle but crucial detail of RUT is the way it computes a max flow on the subnetwork in each iteration. If the max flow computed saturates an arc to some voter unnecessarily, then the result of the algorithm may admit an augmenting path in the main network and thus will fail GRP. Thus, we carefully redistribute flows in each iteration to compute a max flow which only saturates the arc to a voter when strictly necessary. This can be thought of as an intermediate fairness check executed during each iteration of RUT. This redistributive process gives our voting rule its name and will be detailed in the proof of the following technical lemma, which is key to our proof that RUT satisfies GRP.

Lemma 4.2. *Given a network formulation $\mathcal{N}$ and a feasible flow f on $\mathcal{N}(T)$ for some $T \subseteq C$, there exists a polytime computable max flow f' on $\mathcal{N}(T)$ such that each of the following conditions holds:*

1. $\forall i \in N$ *with* $A_i \subseteq T$, $f(s \to i) = \frac{k}{n} \implies f'(s \to i) = \frac{k}{n}$
2. *If the residual network resulting from flow f' on the main network $\mathcal{N}$ admits an augmenting path, then the shortest such augmenting path is of length three.*
3. $f'(c \to t) \geq f(c \to t) \;\; \forall c \in T$

Proof. Starting with f, apply Ford-Fulkerson to compute a max flow f' on $\mathcal{N}(T)$. Since Ford-Fulkerson weakly increases flow on arcs exiting the source, Condition (1) holds so far. We now give a procedure which will give us the desired condition without changing the flow value and thus maintaining the max flow property. The procedure can be thought of as iteratively rebalancing payments in order to avoid fully saturating connections from the source to voters who approve of some candidate not contained in the subnetwork.

Let $N' = \{i \in N : f'(s \to i) = \frac{k}{n}, A_i \setminus T \neq \emptyset\}$, the set of agents whose connection to the source is saturated under f', but still approve of some candidate not in T. Search in the residual network resulting from f' on the subnetwork $\mathcal{N}(T)$ for a cycle $Y_i = (s, i_1, c_1, \ldots, i_r, c_r, i, s)$ with *bottleneck* $b > 0$ (i.e., the minimum residual capacity on the cycle is strictly positive) where $i \in N'$. Update f'

[3]Suppose there are four candidates and three voters with preferences as follws: $A_1 = \{a, b\}, A_2 = \{b, c\}, A_3 = \{d\}$. It can be verified that $\vec{q} = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{3})$ satisfies GRP. Now consider the fractional committee $\vec{p} = (0, \frac{1}{3}, 0, \frac{2}{3})$, which Pareto dominates $\vec{q}$ but does not satisfy GFS with respect to voter group $\{1, 2\}$.

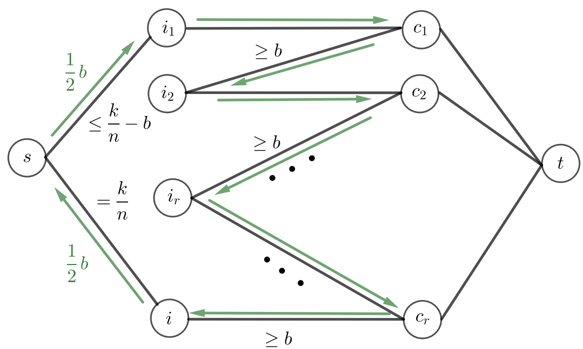

Fig. 2. Illustration of a flow update along a cycle $Y_i = (s, i_1, c_1, \dots, i_r, c_r, i, s)$ with bottleneck $b > 0$. Labels in black denote flow bounds sufficient to trigger the flow update. Green arcs show the path of flow redistribution.

by pushing $\frac{1}{2}b$ additional flow along Y_i. Refer to Figure 2 for an illustration of such a flow update along a cycle. When such a cycle no longer exists, return f'.

Note that the flow value of f' is invariant in this cycle described in the previous paragraph since flow travels in a cycle starting and ending at the source. Thus, f' remains a max flow on $\mathcal{N}(T)$. To see that this procedure is polynomial time computable, observe that identification of a cycle in the residual network with positive bottleneck and ending in an agent in N' can be computed with BFS, and that there are at most n iterations. The latter observation is true since each cycle decreases the size of N' by exactly one. Specifically, after pushing flow along cycle Y_i, $f'(s \to i) = \frac{k}{n} - \frac{1}{2}b < \frac{k}{n}$ (so i no longer belongs to N') and $f'(s \to i_1) \leq (\frac{k}{n} - b) + \frac{1}{2}b < \frac{k}{n}$ (so i_1, the only other voter whose flow from the source changes, is still not in N'). We can see that Condition (1) is retained throughout this procedure by noting that $A_i \not\subseteq T$ since $i \in N'$ and $f(s \to i_1) \leq f'(s \to i_1) < \frac{k}{n}$, and thus the only two voters for whom $f'(s \to i)$ has changed still uphold the condition.

Before proving Condition (2), we first point out that any feasible flow on $\mathcal{N}(T)$ is also feasible on $\mathcal{N}$ by virtue of $\mathcal{N}(T)$ being a subnetwork. This is important as we assume that f' can be treated as a network flow on $\mathcal{N}$ in the statement. We prove Condition (2) by contradiction. Suppose that the shortest augmenting path on the residual network $\mathcal{R}_{f'}$ resulting from network flow f' on $\mathcal{N}$ has a length of four or more. Then, the shortest augmenting path is of the form $P = (s, i_1, c_1, \dots, i_r, c_r, t)$ for some $r \geq 2$. Note that $f'(s \to i_r) = \frac{k}{n}$ since otherwise this edge would have residual capacity and (s, i_r, c_r, t) would constitute an augmenting path of length three. Also note that $c_r \notin T$ since otherwise P would constitute an augmenting path in the residual network from network flow f' on $\mathcal{N}(T)$, contradicting that f' is a max flow on $\mathcal{N}(T)$. Together, these two observations show that $i_r \in N'$ at the termination of the loop.

It is apparent that $\{c_1, \dots, c_{r-1}\} \subseteq T$ since there is residual capacity on the "backward" arc (c_j, i_{j+1}) in $\mathcal{R}_{f'}$ for all $j \in [r-1]$. This means that $Y = (s, i_1, c_1, \dots, i_r, s)$ is a valid cycle in the residual network resulting from flow f' on $\mathcal{N}(T)$. Furthermore, since (i_r, s) has strictly positive residual capacity as does every other arc in Y (by virtue of belonging to an augmenting path P), it holds that the bottleneck of Y is strictly positive. This leads us to a contradiction since the termination condition of our procedure was not met. Condition (3) holds since Ford-Fulkerson weakly increases flows on each arc from a candidate to the sink and the procedure we execute afterwards never alters the flows from a candidate to the sink. □

We can now state and prove the main theorem of the section.

Theorem 4.3. *Redistributive Utilitarian Rule computes a fractional committee satisfying efficiency and group resource proportionality in polynomial time.*

Proof. We first point out that the first while loop (which we will refer to as the main loop) of RUT (Algorithm 1) will always terminate due to the if-condition. This is because, when $j = m$, $\mathcal{N}^j = \mathcal{N}$ and thus applying Lemma 4.2 computes a max flow on $\mathcal{N}$. And if any agent in V^j approved of some candidate that was not integrally funded, there would be an augmenting path. Henceforth, we denote the total number of rounds executed in the main loop as z.

GRP. We now show that $\vec{p}$, the fractional committee computed by RUT, satisfies GRP. Since $\vec{p}$ clearly elementwise dominates the fractional committee given by f^*, it is sufficient by Theorem 3.3 to show that f^* constitutes a max flow on $\mathcal{N}$. Assume, for the sake of a contradiction, that f^* is not a max flow on $\mathcal{N}$. Then, there must be an augmenting path in the residual network $\mathcal{R}_{f^*}$ from network flow f^* on $\mathcal{N}$. However, since f^* was computed by an application of Lemma 4.2, we know that the shortest augmenting path in $\mathcal{R}_{f^*}$ is of length three. That is, there is an augmenting path of the form $P = (s, i, c, t)$ for some $i \in N, c \in C$. Since there must be residual capacity in $\mathcal{R}_{f^*}$ for each arc in P, it is clear that $f^*(s \to i) < cap(s \to i) = \frac{k}{n}$ and $f^*(c \to t) < cap(c \to t) = 1$. However, since (i, c) being an arc in $\mathcal{N}$ implies $c \in A_i$, this means that agent i and candidate c contradict the termination condition of the main loop of RUT.

Efficiency. We will prove efficiency by showing that the fractional committee $\vec{p}$ is equivalent to a greedy selection of candidates in order of their score $s_\lambda(c)$. Let C^+ denote the candidates which receive flow during the main loop of RUT, i.e. $f^*(c \to t) > 0$ for all $c \in C^+$. Let C^- denote the complement set of candidates, i.e. $f^*(c \to t) = 0$ for all $c \in C^-$. It is clear that all candidates in C^+ are included in $\mathcal{N}^z$, the state of the subnetwork when the main loop terminates. By construction, since weights weakly increase over the course of RUT and a candidate c_j is added to the subnetwork $\mathcal{N}^j$ only when it reaches the score $s_\lambda(c_j) = s^*$, we have that $\mathbf{s_\lambda(c) \geq s^*}$ **for all** $\mathbf{c \in C^+}$.

Now suppose there is some candidate c' with $s_\lambda(c') > s^*$. Clearly, since weights only increase during the main loop, $s_\lambda(c')$ hit s^* before the main loop terminated and c' was added to the subnetwork. Thus, $c' = c_j$ for some j. Since the score of c_j continued to increase after this iteration, there must have been some voter i' with $c_j \in A_{i'}$ and $f_{j'}(s \to i') < \frac{k}{n}$ for some $j' \geq j$. Since $f_{j'}$ is computed via iterated application of Lemma 4.2, it is clear from Condition (1) that $f_j(s \to i') < \frac{k}{n}$. Since we know f_j is a max flow on $\mathcal{N}^j$, it must be that $f_j(c' \to t) = 1$, since otherwise (s, i', c', t) would constitute an augmenting path. Then, by Condition (3) of Lemma 4.2, we have that $f^*(c' \to t) = 1$. Note that this observation implies that **for any** c **with** $f^*(c \to t) < 1$, **it must be that** $\mathbf{s_\lambda(c) \leq s^*}$.

We can now show that any two fractionally selected candidates must have equal scores. That is,

$$s_\lambda(c_1) = s_\lambda(c_2) \qquad \forall c_1, c_2 \in C : p_{c_1}, p_{c_2} \in (0, 1) \tag{3}$$

On the contrary, assume without loss of generality, that $s_\lambda(c_1) > s_\lambda(c_2)$. If $c_2 \in C^+$, then

$$s_\lambda(c_1) > s_\lambda(c_2) \geq s^* \implies f^*(c_1 \to t) = 1 > p_{c_1}.$$

Since $p_c \geq f^*(c \to t)$ for all c by construction, this is a contradiction. So it must be that $c_2 \in C^-$ and thus $f^*(c_2 \to t) = 0$, which means that c_2 was fractionally selected during the greedy completion phase (the second while-loop in Algorithm 1). However, this loop selects in order of score, and since c_1 is not integrally selected and has a higher score, we reach a contradiction.

Our second key observation is that the score of any integrally selected candidate is always at least that of any fractionally selected candidate, i.e.

$$s_\lambda(c_1) \geq s_\lambda(c_2) \qquad \forall c_1, c_2 \in C : p_{c_2} < p_{c_1} = 1. \tag{4}$$

To see this, first note that $s_\lambda(c_2) \leq s^*$ by our previous observation since $f^*(c_2 \to t) \leq p_{c_2} < 1$. If $c_1 \in C^+$, then $s_\lambda(c_1) \geq s^*$ and Equation 4 follows. Otherwise, $c_1 \in C^-$ and was selected entirely in the phase which proceeds greedily by score amongst candidates which have not been integrally selected. Since c_2 is not integrally selected, it follows that c_1 has a higher score, i.e. $s_\lambda(c_1) \geq s_\lambda(c_2)$.

Our third and final observation confirms that any candidate which receives positive probability under RUT has a higher score than any candidate which does not, i.e.

$$s_\lambda(c_1) \geq s_\lambda(c_2) \qquad \forall c_1, c_2 \in C : p_{c_1} > p_{c_2} = 0. \tag{5}$$

Note that $c_2 \in C^-$. Thus, Equation 5 follows since $s_\lambda(c_1) \geq s^* \geq s_\lambda(c_2)$ if $c_1 \in C^+$, and by construction of the greedy completion phase if not.

Altogether, from Equations 3-5, we have that $\vec{p}$ is the fractional committee which maximizes the expression

$$\max_{\vec{q}} \sum_{c \in C} q_c s_\lambda(c) = \max_{\vec{q}} \sum_{c \in C} \sum_{i \in N_c} q_c \lambda_i = \max_{\vec{q}} \sum_{i \in N} \sum_{c \in A_i} q_c \lambda_i = \max_{\vec{q}} \sum_{i \in N} \lambda_i u_i(\vec{q}).$$

Thus, $\vec{p}$ maximizes weighted utilitarian welfare for positive weights λ_i, and efficiency follows.

Polynomial time computation. The main loop of RUT adds a candidate to the subnetwork in each iteration and thus terminates in at most m iterations. One can identify the next candidate to add to the subnetwork in polynomial time by simply checking the necessary uniform weight increase for each candidate, and choosing the candidate which requires the minimum weight increase. Computation of the relevant max flow is in polynomial time by Lemma 4.2. The termination condition of the loop is also clearly polynomial time checkable. Lastly, the greedy completion step requires only a linear pass over a sorted list of the candidates. □

We note that, when applied to the $k = 1$ special case, RUT is equivalent to the *fair utilitarian rule* defined by Bogomolnaia et al. [8]. However, in the single-winner context, each voter allocates their entire share in a single round, and thus the algorithm is significantly simpler and it is not necessary to use a network flow-based approach. Brandl et al. [10] generalized the fair utilitarian rule to the single-winner setting with arbitrary voter endowments. We note that RUT could similarly be extended to a generalization of our setting with endowments.

4.2 On Strategyproofness and GRP-efficiency

While RUT is both efficient and fair, it is not strategyproof. This is inevitable in light of Theorem 2 from Brandl et al. [10] which states that no strategyproof and efficient rule can satisfy *positive share* – a minimal fairness requirement which guarantees every voter non-zero utility. In the single-winner setting, when strategyproofness and fairness are viewed as strict requirements, the most attractive voting rule is the *conditional utilitarian rule* (CUT). For each voter i, the rule distributes $1/n$ to the candidates in A_i which are approved by the greatest number of voters. CUT is strategyproof and maximizes utilitarian welfare subject to GFS. However, as we will now show, any extension of CUT to probabilistic committee voting must lose one of these properties.

We say a rule satisfies *GRP-efficiency* (or is *GRP-efficient*) if it always returns a fractional committee which is efficient among fractional committees that satisfy GRP. We define *GFS-efficiency* analagously.

ALGORITHM 2: Generalized CUT

Input: Voters N, candidates C, approval profile $(A_i)_{i \in N}$ and committee size k.
Output: A fractional committee $\vec{p} = (\Delta_c)_{c \in C}$ of size k.

Let $U = (C_1, C_2, \dots C_l)$ be a partition of C ordered by decreasing approval score, i.e. for $a \in C_i, b \in C_j$, $|N_a| > |N_b| \iff i < j$ and $|N_a| = |N_b| \iff i = j$.
Add costs to $\mathcal{N}$:

- $cost(c, t) = r$ for all $c \in C_r, r \in [l]$
- $cost(u, v)$ for all other arcs in $\mathcal{N}$

Compute a minimum-cost maximum-flow f^* on $\mathcal{N}$.
Let $\vec{p}^0$ be the fractional committee given by f^*.
$\vec{p} \leftarrow \vec{p}^0$.
Complete $\vec{p}$ greedily in order of U until $\sum_{c \in C} p_c = k$.
return $\vec{p}$

Proposition 4.4. *No voting rule satisfies GRP-efficiency and strategyproofness. Likewise, no voting rule satisfies GFS-efficiency and strategyproofness.*

Proof. Consider an instance with $m = n + 1$, committee size $k = 2$, and an approval profile $\mathcal{A}$ defined as follows: $A_i = \{c_0, c_i\}$ for all $i \in N$. Suppose there is a rule F satisfying GRP-efficiency. Let $\vec{p} = F(\mathcal{A}, k)$ be the fractional committee returned by the rule. Note that, since $k = 2$, it must be that $p_{c_j} > 0$ for some $j \in [n]$. If $p_{c_0} < 1$, then we can shift $\epsilon = \min(p_{c_j}, 1 - p_{c_0})$ probability to p_{c_0} from p_{c_j} without losing GRP. This holds since c_0 is approved by every voter and thus the LHS of every GRP constraint is unchanged by this shift. Call the resulting fractional committee $\vec{q}$. It is apparent that $u_j(\vec{q}) = u_j(\vec{q})$ and $u_i(\vec{q}) = u_i(\vec{q})$ for all $i \neq j$. Thus, $\vec{q}$ Pareto dominates $\vec{p}$. This contradicts GRP-efficiency, and thus we can conclude that $p_{c_0} = 1$.

Since $\sum_{r=1}^{n} p_{c_r} = k - p_{c_0} = 1$, it follows that $p_{c_j} \leq \frac{1}{n}$ for some $j \in [n]$. Note that $u_j(\vec{p}) = p_{c_0} + p_{c_j} \leq 1 + \frac{1}{n}$. Now consider an alternative approval profile where voter j misreports her preferences by dropping c_0, i.e. the approval profile $\mathcal{A}' = \{A_1, A_2, \dots, A'_j, \dots, A_n\}$ where $A'_j = \{c_j\}$. Let $\vec{q} = F(\mathcal{A}', k)$ be the fractional committee returned by the rule in this case. By a similar argument to before, it must be that $q_{c_0} = 1$. Furthermore, GRP requires that $\sum_{c \in A'_j} q_c \geq \frac{k}{n} = \frac{2}{n}$. Thus,

$$u_j(F(\mathcal{A}', k)) = u_j(\vec{q}) = q_{c_0} + q_{c_j} \geq 1 + \frac{2}{n} > 1 + \frac{1}{n} \geq u_j(\vec{p}) = u_j(F(\mathcal{A}, k)).$$

This proves that F is not strategyproof. The same argument can be used to show that GFS-efficiency and strategyproofness are likewise incompatible. □

Due to the impossibility, we are forced to compromise when designing probabilistic committee voting rules. It is known that the *random dictator* family of voting rules satisfy GFS and strategyproofness in probabilistic committee voting [6]. We will introduce a new rule, *Generalized CUT*, which maximizes utilitarian welfare subject to GRP (and is thus GRP-efficient).[4]

At a high level, the rule computes a max flow, while prioritizing candidates in order of their contribution to social welfare. To do so, it reduces the instance to a minimum-cost maximum-flow problem. Formally, let $U = (C_1, C_2, \dots C_l)$ be a partition of C ordered by decreasing approval score, i.e. for $a \in C_i, b \in C_j$, $|N_a| > |N_b| \iff i < j$ and $|N_a| = |N_b| \iff i = j$. We set the edge

[4]We note that welfare maximization is arguably a more natural objective than weighted welfare maximization according to the somewhat artificial weights computed by RUT. Thus, while RUT satisfies efficiency and GRP, it is not the case that it is always prefereable to Generalized CUT.

costs to be such that $cost(c,t) = r$ for all $c \in C_r, r \in [l]$, and $cost(u,v) = 0$ for all other arcs in $\mathcal{N}$. Generalized CUT first computes a minimum-cost maximum-flow solution f^*.[5] The algorithm then allocates the remaining $k - val(f^*)$ probability to candidates in order of the candidate partition $\mathcal{U}$. Refer to Algorithm 2 for pseudocode. We now prove that Generalized CUT maximizes utilitarian welfare subject to GRP.

Proposition 4.5. *Generalized CUT maximizes utilitarian welfare subject to GRP.*

Proof. Let $\vec{p}$ be the fractional committee returned by Generalized CUT. Since $\vec{p}$ clearly element-wise dominates the fractional committee given by f^*, a max flow on $\mathcal{N}$ by virtue of being a solution to the minimum-cost maximum-flow problem on $\mathcal{N}$ (with costs added), it follows from Theorem 3.3 that $\vec{p}$ satisfies GRP.

We must now show that $\vec{p}$ maximizes utilitarian welfare amongst GRP outcomes. Let $\vec{q}$ denote some fractional committee which satisfies GRP. Our goal is to show that the social welfare attained by $\vec{p}$ is at least that of $\vec{q}$, i.e. $\sum_{i\in N} u_i(\vec{p}) \geq \sum_{i\in N} u_i(\vec{q})$. We denote by $\vec{q}^0$ the fractional committee given by some max flow f' on $\mathcal{N}$ such that $q_c \geq q_c^0$ for all $c \in C$. Note that such a fractional committee must exist by the characterization of GRP (Theorem 3.3), which $\vec{q}$ satisfies. Recall that $\vec{p}^0$ is the fractional committee given by f^*. We also introduce some new notation. Let $\mathcal{U} : Q^m \to Q^l$ provide a mapping from fractional committees to a vector giving the aggregate amount of probability allocated to each set of candidates in the partition $\mathcal{U}$. That is, for some fractional committee $\vec{x}$, $\mathcal{U}(\vec{x})_j = \sum_{c\in C_j} x_c$ for all $j \in [l]$.

Claim 1. $\sum_{r\in[j]} \mathcal{U}(\vec{p}^0)_r \geq \sum_{r\in[j]} \mathcal{U}(\vec{q}^0)_r$ for all $j \in [l]$.
Consider the network $\mathcal{N}^j = \mathcal{N}(\bigcup_{r\in[j]} C_r)$, which is the same as the original network, but with all candidates in $\{C_{j+1}, \ldots, C_l\}$ and associated edges deleted. Let f_j be a flow on $\mathcal{N}^j$ defined as follows:

- $f_j(c \to t) = f^*(c \to t)$ for all $c \in \bigcup_{r\in[j]} C_r$
- $f_j(i \to c) = f^*(i \to c)$ for all $(i,c) \in \mathcal{N}^j$
- $f_j(s \to i) = \sum_{c\in\bigcup_{r\in[j]} C_r} f_j(i \to c)$ for all $i \in N$

By construction, f_j respects capacities and conservation of flows and is thus a feasible flow on $\mathcal{N}^j$. We claim that f_j is a max flow on $\mathcal{N}^j$. Suppose for a contradiction that there is an augmenting path $P = (s, i_1, c_1, i_2, c_2, \ldots, i_h, c_h, t)$ on the residual network of f_j. Note that if $f^*(s \to i_1) < cap(s \to i_1) = \frac{k}{n}$, then P is an augmenting path on the residual network of f^* and this contradicts that f^* is a max flow on $\mathcal{N}$. Thus, we assume that $f^*(s \to i_1) = \frac{k}{n}$. Then,

$$\frac{k}{n} = \sum_{c\in C} f^*(i_1 \to c) = \sum_{c\in\bigcup_{r\in[j]} C_r} f^*(i_1 \to c) + \sum_{c\in\bigcup_{r=j+1}^{l} C_r} f^*(i_1 \to c) = \sum_{c\in\bigcup_{r\in[j]} C_r} f_j(i_1 \to c) + \sum_{c\in\bigcup_{r=j+1}^{l} C_r} f^*(i_1 \to c)$$
$$= f_j(s \to i_1) + \sum_{c\in\bigcup_{r=j+1}^{l} C_r} f^*(i_1 \to c) < \frac{k}{n} + \sum_{c\in\bigcup_{r=j+1}^{l} C_r} f^*(i_1 \to c).$$

where the inequality follows since (s, i_1) is an arc in P and thus $f_j(s \to i_1) < cap(s, i_1)$. This tells us that $f^*(i_1 \to c') > 0$ for some $c' \in \bigcup_{r=j+1}^{l} C_r$.

Now consider the cycle $Y = (t, c', i_1, c_1, i_2, c_2, \ldots, i_h, c_h, t)$ in the residual network $\mathcal{R}_{f^*}$ of f^*. It is clear that Y has strictly positive residual capacity. Furthermore, because $c_h \in \bigcup_{r\in[j]} C_r$ and $c' \in \bigcup_{r=j+1}^{l} C_r$, we know that $cost(c' \to t) > cost(c_h \to t)$, and thus Y is a negative cost cycle. That is, by pushing flow along Y, we can construct a network flow with equal flow value to that of f^* and

[5]This step can be carried out with any polynomial time method for solving minimum-cost maximum-flow problems, e.g. Cut Canceling [23, 30], guaranteeing that Generalized CUT runs in polynomial time.

strictly lower cost. However, this contradicts that f^* is the optimal solution to the minimum-cost maximum-flow problem.

This leads to the statement of our claim:

$$\sum_{r\in[j]} \mathcal{U}(\vec{p}^0)_r = \sum_{c\in\bigcup_{r\in[j]} C_r} p_c^0 = \sum_{c\in\bigcup_{r\in[j]} C_r} f^*(c\to t)$$
$$\geq \sum_{c\in\bigcup_{r\in[j]} C_r} f'(c\to t) = \sum_{c\in\bigcup_{r\in[j]} C_r} q_c^0 = \sum_{r\in[j]} \mathcal{U}(\vec{p}^0)_r$$

where the inequality follows since otherwise f' would correspond to a flow of greater value on $\mathcal{N}^j$ using an analagous mapping to that used to construct f_j.

We can now use Claim 1 to show that $\mathcal{U}(\vec{p})$ stochastically dominates $\mathcal{U}(\vec{q})$, i.e., $\sum_{r\in[j]} \mathcal{U}(\vec{p})_r \geq \sum_{r\in[j]} \mathcal{U}(\vec{q})_r$ for all $j \in [l]$. Fix $j \in [l]$. Observe that $\vec{p}$ is computed by allocating $k - \sum_{c\in C} p_c^0$ probability to candidates in the order of the candidate partition $\mathcal{U}$. Thus, a candidate in $C_{j'}$ for $j' > j$ is allocated probability in this "greedy step" only if $p_c = 1$ for all $c \in \cup_{r\in[j]} C_j$. Taken altogether, this leads us to our stochastic dominance relation:

$$\sum_{r\in[j]} \mathcal{U}(\vec{p}) = \sum_{r\in[j]}\sum_{c\in C_r} p_c = \min\left\{\sum_{r\in[j]} |C_r|, \sum_{r\in[j]}\sum_{c\in C_r} p_c^0 + (k - \sum_{c\in C} p_c^0)\right\}$$
$$\geq \min\left\{\sum_{r\in[j]} |C_r|, \sum_{r\in[j]}\sum_{c\in C_r} q_c^0 + (k - \sum_{c\in C} p_c^0)\right\} \quad \because \text{ Claim 1}$$
$$= \min\left\{\sum_{r\in[j]} |C_r|, \sum_{r\in[j]}\sum_{c\in C_r} q_c^0 + (k - \sum_{c\in C} q_c^0)\right\} \quad \because f^* \text{ and } f' \text{ are both max flows}$$
$$= \min\left\{\sum_{r\in[j]} |C_r|, k - \sum_{r=j+1}^{l}\sum_{c\in C_r} q_c^0\right\}$$
$$\geq \min\left\{\sum_{r\in[j]} |C_r|, k - \sum_{r=j+1}^{l}\sum_{c\in C_r} q_c\right\} \quad \because q_c \geq q_c^0 \quad \forall c \in C$$
$$= \sum_{r\in[j]}\sum_{c\in C_r} q_c \quad \because \sum_{c\in C} q_c = k \text{ and } q_c \leq 1 \quad \forall c\in C$$
$$= \sum_{r\in[j]} \mathcal{U}(\vec{q}).$$

Let $\mathbb{1}_{\{\cdot\}}$ be an indicator function and let a_j denote the approval score of each candidate in C_j, i.e. $a_j = |N_c|, c \in C_j$. We now have what we need to show that $\vec{p}$ attains a greater social welfare than $\vec{q}$:

$$\sum_{i\in N} u_i(\vec{p}) - \sum_{i\in N} u_i(\vec{q}) = \sum_{i\in N}\sum_{c\in C} p_c \mathbb{1}_{\{c\in A_i\}} - \sum_{i\in N}\sum_{c\in C} q_c \mathbb{1}_{\{c\in A_i\}}$$
$$= \sum_{c\in C} |N_c| \cdot (p_c - q_c) \quad = \quad \sum_{j\in[l]}\sum_{c\in C_r} a_j \cdot (p_c - q_c)$$
$$= \sum_{j\in[l]} a_j \cdot [\mathcal{U}(\vec{p})_j - \mathcal{U}(\vec{q})_j] \ \geq \ 0$$

The final inequality follows from Lemma 4.6 once we note that $\{a_j\}_{j\in[l]}$ is non-increasing and the sequence $\{\mathcal{U}(\vec{p})_j - \mathcal{U}(\vec{q})_j\}_{j\in[l]}$ has non-negative partial sums because $\{\mathcal{U}(\vec{p})\}$ stochastically dominates $\{\mathcal{U}(\vec{q})\}$.

□

Lemma 4.6. *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of r real numbers such that $a_1 \geq a_2 \geq \ldots \geq a_r \geq 0$ and for all $r' \in [r]$, it holds that $\sum_{i\in[r']} b_i \geq 0$. Then, it must be that $\sum_{i\in[r]} a_i \cdot b_i \geq 0$.*

5 BEST-OF-BOTH-WORLDS FAIRNESS

In this section, we will build upon the characterization of our new axiom to obtain a general *best-of-both-worlds* result. This result leads us to strengthen two known compatibility results from Aziz et al. [6], answering an open question left by the authors in the process. We begin with a lemma which, at a high level, shows that fractional committee which can be represented by a network flow can be completed to a max flow.

Lemma 5.1. *Let $\vec{q}$ be the committee given by a feasible flow on network representation $\mathcal{N}$. There exists a max flow f^* on $\mathcal{N}$, computable in polynomial time, such that the committee $\vec{p}$ given by f^* elementwise dominates $\vec{q}$.*

Proof. Let f be a feasible flow on network representation $\mathcal{N}$ and $\mathcal{R}_f$ be the residual network resulting from f. Also let $\vec{q}$ be the fractional committee given by f. If there is no augmenting path on $\mathcal{R}_f$, then f is a max flow and the statement holds since $\vec{q}$ elementwise dominates $\vec{q}$. Otherwise, if there is some augmenting path, send flow along this path, and denote the resulting flow f'. The final arc in the augmenting path must be of the form (c', t) for some $c' \in C$ by the structure of $\mathcal{N}$. Since (c', t) is an arc in $\mathcal{N}$, it is clear that $f'(c' \to t) > f(c' \to t)$. Note that the augmenting path can only contain one arc entering the sink (since otherwise it would contain a cycle), and thus $f'(c \to t) = f(c \to t)$ for all $c \in C \setminus \{c'\}$. It is now apparent that the fractional committee given by f' elementwise dominates $\vec{q}$. Thus, this property can be maintained while eliminating augmenting paths, which will lead to a max flow in polynomial time. □

The results of this section make use of a property which Brill and Peters (2024) have termed *affordability*, which weakens the definition of priceability from Peters and Skowron (2020) by removing the final condition that no unselected candidate can be afforded by its supporters.

Definition 5.2 (Affordability)**.** We say W is *affordable* if there exists a payment function $\pi_i : C \to \mathbb{R}_{\geq 0}$ satisfying the following four conditions:

(1) $\pi_i(c) = 0$ for each $i \in N$, $c \notin A_i$
(2) $\sum_{c\in C} \pi_i(c) \leq k/n$ for each $i \in N$
(3) $\sum_{i\in N} \pi_i(c) = 1$ for each $c \in W$
(4) $\sum_{i\in N} \pi_i(c) = 0$ for each $c \notin W$.

The main theorem of this section shows that, for every affordable committee W, there is an ex-ante GRP lottery for which every committee in its support contains W.

Theorem 5.3. *Let W be an affordable committee. There exists a lottery $\Delta = \{(\lambda_j, W_j)\}_{j\in[s]}$ such that (1) Δ satisfies GRP and (2) $W \subseteq W_j$ for all $j \in [s]$. Furthermore, given W, such a lottery can be computed in polynomial time.*

Proof. For some instance of our problem, let $\mathcal{N}$ be the network formulation and W be some affordable committee.

Now consider the flow f on $\mathcal{N}$ defined as follows:

- $f(s,i) = \sum_{j \in C} \pi_i(c)$ for all $i \in N$
- $f(i,c) = \pi_i(c)$ for all $i \in N, c \in C$
- $f(c,t) = \sum_{i \in N} \pi_i(c)$ for all $c \in C$.

We claim that f is a feasible flow on $\mathcal{N}$. It is apparent that f respects conservation of flows as each voter's incoming flow is equal to their outgoing flow and the same is true for candidate nodes. To see that the capacity constraints of $\mathcal{N}$ are respected by f, note that Condition 2 ensures that $f(s,i) \leq k/n = cap(s,i)$ for all $i \in N$, Condition 1 ensures that $f(i,c) = 0 = cap(i,c)$ for all $i \in N, c \notin A_i$, and Conditions 3 and 4 ensure that $f(c,t) \leq 1 = cap(c,t)$ for all $c \in C$.

Let $\vec{q}$ be the committee given by f on $\mathcal{N}$. Since f is feasible, we can conclude by Lemma 5.1 that there exists a max flow f^* such that the committee $\vec{p}$ given by f^* elementwise dominates $\vec{q}$. Consider some implementation Δ of $\vec{p}$. Since $p_c \geq q_c = f(c \rightarrow t) = 1$ for all $c \in W$ by Condition 3, we have that every committee in the support of Δ contains W. Finally, because $\vec{p}$ is the fractional committee given by the max flow f on $\mathcal{N}$, we have by Theorem 3.3 that $\vec{p}$ and thus Δ satisfies GRP. □

Pierczynski et al. (2021) gave an exponential-time algorithm for an FJR committee and showed (Lemma 2) that this committee always satisfies affordability. This leads us to the following corollary to Theorem 5.3, which resolves an open question posed by Aziz et al. [6].

Corollary 5.4. *A lottery satisfying ex-post FJR and ex-ante GRP is guaranteed to exist.*

Aziz et al. [6] also gave an algorithm using the *Method of Equal Shares* (MES) [39] as a subroutine and showed that this algorithm gives EJR+, Strong UFS, and GFS. Since MES always returns a priceable (and thus affordable) committee, another consequence of Theorem 5.3 is a strengthening of Theorem 4.1 from Aziz et al. [6]. By applying the procedure described in the proof of Lemma 5.1 to the result of MES, we can obtain a randomized committee which obtains all of the ex-post properties of MES in tandem with GRP.[6]

Corollary 5.5. *A lottery satisfying ex-post EJR+ and ex-ante GRP can be computed in polynomial time.*

As these corollaries show, Theorem 5.3 provides a useful tool that could aid in producing further best-of-both-worlds results in public decision problems.

DISCUSSION

In this paper, we introduced a new fairness axiom for probabilistic committee voting. We characterized our axiom and gave several algorithmic results exploring compatibility with efficiency, strategyproofness, and ex-post fairness. Our characterization demonstrated a connection between fair committee voting and network flows. We believe that this network-based approach will prove useful for extensions of committee voting such as participatory budgeting or the setting with arbitrary voter entitlements.

In Section 4.2, we showed that strategyproofness is incompatible with GRP-efficiency and GFS-efficiency (Proposition 4.4). One way of circumventing this impossibility is by weakening strategyproofness. A natural choice may be *excludable strategyproofness*, introduced in the single-winner setting by Aziz et al. [1], which guarantees no voter has incentive to deviate if they can only derive utility from candidates they claimed to approve. It would be interesting to explore the

[6]Note that rules belonging to BW-MES, the family of rules which give ex-post EJR+, ex-ante GFS, and ex-ante Strong UFS (Theorem 4.1, Aziz et al. [6]), do not necessarily satisfy GRP. To see this, suppose one voter with leftover budget after the MES phase does not approve of any unselected candidate, while another voter has no budget remaining and approves some unselected candidate. An arbitrary allocation of the remaining budget need not satisfy GRP with respect to two such voters.

compatibility of excludable strategyproofness with the various notions of fairness and efficiency studied in this paper.

Proposition 4.4 puts fair, (weakly) efficient, and strategyproof rules even further from reach in the probabilistic committee voting setting than in the single-winner setting. However, plenty of questions remain surrounding the compatibility of strategyproofness and fairness in our setting. As we pointed out, random dictator satisfies strategyproof and GFS. However, to our knowledge, there is no known strategyproof rule that satisfies Strong UFS (or even the weaker Strong IFS). Since GRP implies Strong UFS, one would need to tackle this issue in order to prove compatibility of GRP and strategyproofness. Our definition of strategyproofness can also be rewritten for rules which return lotteries, in which case it embodies *ex-ante* strategyproofness. We believe studying ex-ante strategyproofness is one of the most intriguing directions for future work in best of both worlds committee voting, particularly because *ex-post* strategyproofness has shown to be incompatible with even a very weak form of ex-post fairness [38].

REFERENCES

[1] Haris Aziz, Anna Bogomolnaia, and Hervé Moulin. 2019. Fair Mixing: The Case of Dichotomous Preferences. In *Proceedings of the 2019 ACM Conference on Economics and Computation*. 753–781.

[2] Haris Aziz, Felix Brandt, Edith Elkind, and Piotr Skowron. 2019. Computational social choice: The first ten years and beyond. *Computing and Software Science: State of the Art and Perspectives* (2019), 48–65.

[3] Haris Aziz, Markus Brill, Vincent Conitzer, Edith Elkind, Rupert Freeman, and Toby Walsh. 2017. Justified Representation in Approval-based Committee Voting. *Social Choice and Welfare* 48, 2 (2017), 461–485.

[4] Haris Aziz, Edith Elkind, Shenwei Huang, Martin Lackner, Luis Sanchez-Fernandez, and Piotr Skowron. 2018. On the Complexity of Extended and Proportional Justified Representation. *Proceedings of the AAAI Conference on Artificial Intelligence* 32, 1 (April 2018).

[5] Haris Aziz, Omer Lev, Nicholas Mattei, Jeffrey S. Rosenschein, and Toby Walsh. 2019. Strategyproof Peer Selection using Randomization, Partitioning, and Apportionment. *Artificial Intelligence* 275 (2019), 295–309.

[6] Haris Aziz, Xinhang Lu, Mashbat Suzuki, Jeremy Vollen, and Toby Walsh. 2023. Best-of-Both-Worlds Fairness in Committee Voting. *Proceedings of the 19th Conference on Web and Internet Economics (WINE)* (2023).

[7] Haris Aziz, Xinhang Lu, Mashbat Suzuki, Jeremy Vollen, and Toby Walsh. 2023. Fair Lotteries for Participatory Budgeting. *Proceedings of the 37th AAAI Conference on Artificial Intelligence (AAAI)* (2023).

[8] Anna Bogomolnaia, Hervé Moulin, and Richard Stong. 2002. Collective Choice under Dichotomous Preferences. (2002). http://www.ucl.ac.uk/~uctpcab/jocs/moulin.pdf.

[9] Anna Bogomolnaia, Hervé Moulin, and Richard Stong. 2005. Collective Choice under Dichotomous Preferences. *Journal of Economic Theory* 122, 2 (June 2005), 165–184.

[10] Florian Brandl, Felix Brandt, Dominik Peters, and Christian Stricker. 2021. Distribution Rules Under Dichotomous Preferences: Two Out of Three Ain't Bad. In *Proceedings of the 22nd ACM Conference on Economics and Computation*. 158–179.

[11] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jerome Lang, and Ariel D Procaccia. 2016. Handbook of Computational Social Choice. (2016), 554.

[12] Markus Brill, Rupert Freeman, Svante Janson, and Martin Lackner. 2023. Phragmén's Voting Methods and Justified Representation. *Mathematical Programming* (2023).

[13] Markus Brill and Jannik Peters. 2023. Robust and Verifiable Proportionality Axioms for Multiwinner Voting. In *Proceedings of the 24th ACM Conference on Economics and Computation, EC 2023, London, United Kingdom, July 9-12, 2023*. ACM, 301. https://doi.org/10.1145/3580507.3597785

[14] Jeff Burdges, Alfonso Cevallos, Peter Czaban, Rob Habermeier, Syed Hosseini, Fabio Lama, Handan Kilinç Alper, Ximin Luo, Fatemeh Shirazi, Alistair Stewart, and Gavin Wood. 2020. Overview of Polkadot and its Design Considerations. *IACR Cryptol. ePrint Arch.* (2020), 641. https://eprint.iacr.org/2020/641

[15] Ozan Candogan, Markos Epitropou, and Rakesh V. Vohra. 2016. Competitive Equilibrium and Trading Networks: A Network Flow Approach. In *Proceedings of the 2016 ACM Conference on Economics and Computation, EC '16, Maastricht, The Netherlands, July 24-28, 2016*. ACM, 701–702. https://doi.org/10.1145/2940716.2940782

[16] Abhijnan Chakraborty, Gourab K Patro, Niloy Ganguly, Krishna P Gummadi, and Patrick Loiseau. 2019. Equality of voice: Towards fair representation in crowdsourced top-k recommendations. In *Proceedings of the Conference on Fairness, Accountability, and Transparency*. 129–138.

[17] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. 2010. Dependent Randomized Rounding via Exchange Properties of Combinatorial Structures. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*. IEEE Computer Society, 575–584. https://doi.org/10.1109/FOCS.2010.60

[18] Yu Cheng, Zhihao Jiang, Kamesh Munagala, and Kangning Wang. 2020. Group Fairness in Committee Selection. *ACM Transactions on Economics and Computation (TEAC)* 8, 4 (2020), 1–18.

[19] Henry Richmond Droop. 1881. On methods of electing representatives. *Journal of the Statistical Society of London* 44, 2 (1881), 141–202.

[20] Conal Duddy. 2015. Fair Sharing under Dichotomous Preferences. *Mathematical Social Sciences* 73 (2015), 1–5.

[21] Edith Elkind, Piotr Faliszewski, Ayumi Igarashi, Pasin Manurangsi, Ulrike Schmidt-Kraepelin, and Warut Suksompong. 2022. Justifying Groups in Multiwinner Approval Voting. arXiv:2108.12949 [cs]

[22] Ulle Endriss. 2017. *Trends in Computational Social Choice*.

[23] Thomas R Ervolina and S Thomas McCormick. 1993. Two strongly polynomial cut cancelling algorithms for minimum cost network flow. *Discrete Applied Mathematics* 46, 2 (1993), 133–165.

[24] Piotr Faliszewski, Jakub Sawicki, Robert Schaefer, and Maciej Smolka. 2017. Multiwinner Voting in Genetic Algorithms. *IEEE Intell. Syst.* 32, 1 (2017), 40–48. https://doi.org/10.1109/MIS.2017.5

[25] Michal Feldman, Simon Mauras, Vishnu V. Narayan, and Tomasz Ponitka. 2023. Breaking the Envy Cycle: Best-of-Both-Worlds Guarantees for Subadditive Valuations. *CoRR* abs/2304.03706v1 (2023).

[26] Rupert Freeman, Nisarg Shah, and Rohit Vaish. 2020. Best of Both Worlds: Ex-Ante and Ex-Post Fairness in Resource Allocation. In *Proceedings of the 21st ACM Conference on Economics and Computation (EC)*. 21–22.

[27] Rajiv Gandhi, Samir Khuller, Srinivasan Parthasarathy, and Aravind Srinivasan. 2006. Dependent Rounding and Its Applications to Approximation Algorithms. *J. ACM* 53, 3 (2006), 324–360.

[28] Grzegorz Gawron and Piotr Faliszewski. 2022. Using multiwinner voting to search for movies. In *European Conference on Multi-Agent Systems*. Springer, 134–151.

[29] Allan Gibbard. 1977. Manipulation of schemes that mix voting with chance. *Econometrica: Journal of the Econometric Society* (1977), 665–681.

[30] Andrew V Goldberg and Robert E Tarjan. 1989. Finding minimum-cost circulations by canceling negative cycles. *Journal of the ACM (JACM)* 36, 4 (1989), 873–886.

[31] Martin Hoefer, Marco Schmalhofer, and Giovanna Varricchio. 2023. Best of Both Worlds: Agents with Entitlements. In *Proceedings of the 22nd International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*. 564–572.

[32] Jonas Israel and Markus Brill. 2021. Dynamic Proportional Rankings. In *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021*. ijcai.org, 261–267. https://doi.org/10.24963/ijcai.2021/37

[33] Martin Lackner and Piotr Skowron. 2022. Approval-based committee voting. In *Multi-Winner Voting with Approval Preferences*. Springer, 1–7.

[34] Martin Lackner and Piotr Skowron. 2023. *Multi-Winner Voting with Approval Preferences*. Springer International Publishing, Cham.

[35] Erik Lindahl. 1958. Just taxation—a positive solution. In *Classics in the theory of public finance*. Springer, 168–176.

[36] Marcin Michorzewski, Dominik Peters, and Piotr Skowron. 2020. Price of Fairness in Budget Division and Probabilistic Social Choice. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 34. 2184–2191.

[37] Kamesh Munagala, Yiheng Shen, Kangning Wang, and Zhiyi Wang. 2022. Approximate Core for Committee Selection via Multilinear Extension and Market Clearing. In *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*. SIAM, 2229–2252. https://doi.org/10.1137/1.9781611977073.89

[38] Dominik Peters. 2021. Proportionality and Strategyproofness in Multiwinner Elections. *arXiv:2104.08594 [cs]* (April 2021). arXiv:2104.08594 [cs]

[39] Dominik Peters, Grzegorz Pierczynski, and Piotr Skowron. [n. d.]. Proportional Participatory Budgeting with Cardinal Utilities. ([n. d.]), 25.

[40] Dominik Peters and Piotr Skowron. 2020. Proportionality and the Limits of Welfarism. In *Proceedings of the 21st ACM Conference on Economics and Computation*. 793–794.

[41] Luis Sánchez-Fernández, Edith Elkind, Martin Lackner, Norberto Fernández, Jesús Fisteus, Pablo Basanta Val, and Piotr Skowron. 2017. Proportional Justified Representation. In *Proceedings of the AAAI Conference on Artificial Intelligence*, Vol. 31.

[42] Piotr Skowron, Martin Lackner, Markus Brill, Dominik Peters, and Edith Elkind. 2017. Proportional Rankings. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*. ijcai.org, 409–415. https://doi.org/10.24963/ijcai.2017/58

[43] Aravind Srinivasan. 2001. Distributions on level-sets with applications to approximation algorithms. In *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*. 588–597.

[44] Vijay V Vazirani. 2007. Combinatorial algorithms for market equilibria. *Algorithmic game theory* (2007), 103–134.

PROOF OF PROPOSITION 3.6

Proposition 3.6. *Group resource proportionality implies Strong UFS and GFS.*

Proof. We first show that GRP implies Strong UFS. Consider any group of voters $S \subseteq N$ with $A_i = A_j$ for any $i, j \in S$. For any voter $i \in S$, we have that

$$
\begin{aligned}
u_i(\vec{p}) = \sum_{c \in A_i} p_c &\geq |S|\frac{k}{n} - \max_{T \subseteq S}\left[|T|\frac{k}{n} - |\bigcup_{j \in T} A_j|\right] && \because \vec{p} \text{ satisfies GRP} \\
&= |S|\frac{k}{n} - \max_{T=\emptyset, T=S}\left[|T|\frac{k}{n} - |\bigcup_{j \in T} A_j|\right] && \because \bigcup_{j \in T} A_j = A_i \text{ for any } T \neq \emptyset \\
&= |S|\frac{k}{n} - \max\left[|S|\frac{k}{n} - |A_i|,\ 0\right] \\
&= \min\left[|S|\frac{k}{n},\ |A_i|\right]
\end{aligned}
$$

Next we show that GRP implies GFS. Fix arbitrary set of voters $S \subseteq N$. Let $Q = \{i \in S \mid |A_i| \leq k\}$ and $T^* = \arg\max_{T \subseteq S} \left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right]$, observe that $T^* \subseteq Q$. To see this suppose there exists $j \in S \setminus Q$ (and thus $|A_j| \geq k+1$) such that $j \in T^*$. We have that $|\bigcup_{i \in T^*} A_i| \geq k+1$ but this results in contradiction as $0 \leq |T^*|\frac{k}{n} - |\bigcup_{i \in T^*} A_i| \leq |T^*|\frac{k}{n} - (k+1) < 0$. Finally, we see that,

$$
\begin{aligned}
\sum_{c \in \bigcup_{i \in S} A_i} p_i &\geq |S|\frac{k}{n} - \max_{T \subseteq S}\left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right] && \because \vec{p} \text{ satisfies GRP} \\
&= |S|\frac{k}{n} - \max_{T \subseteq Q}\left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right] \\
&\geq |S|\frac{k}{n} - \max_{T \subseteq Q}\left[|T|\frac{k}{n} - \frac{1}{n}\sum_{i \in T}|A_i|\right] && \because |\bigcup_{i \in T} A_i| \geq \frac{1}{n}\sum_{i \in T}|A_i| \\
&= |S|\frac{k}{n} - \left[|Q|\frac{k}{n} - \frac{1}{n}\sum_{i \in Q}|A_i|\right] \\
&= |S \setminus Q|\frac{k}{n} + \frac{1}{n}\sum_{i \in Q}|A_i| \ = \ \frac{1}{n}\sum_{i \in S}\min(k, |A_i|)
\end{aligned}
$$

□

PROOF OF PROPOSITION 3.7

Definition (PJR [41]). For any positive integer ℓ, a group of voters $N' \subseteq N$ is said to be *ℓ-cohesive* if $|N'| \geq \ell \cdot n/k$ and $|\bigcap_{i \in N'} A_i| \geq \ell$. A committee W is said to satisfy *proportional justified representation (PJR)* if for every positive integer ℓ and every ℓ-cohesive group of voters $N' \subseteq N$, it holds that $|(\bigcup_{i \in N'} A_i) \cap W| \geq \ell$.

Proposition 3.7. *If an integral committee satisfies GRP, then it satisfies PJR.*

Proof. Suppose, for some integral committee W, $\vec{p} = \vec{1}_W$ satisfies GRP. Let $N' \subseteq N$ be some ℓ-cohesive group for some positive integer ℓ. Since $\vec{p}$ satisfies GRP, we have

$$\left|W \cap \bigcup_{i \in N'} A_i\right| = \sum_{c \in \bigcup_{i \in N'} A_i} p_c \geq |N'|\frac{k}{n} - \max_{T \subseteq N'} \left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right]$$
$$\geq |N'|\frac{k}{n} - \max_{T \subseteq N'} \left[|T|\frac{k}{n} - \ell\right] = \ell$$

where the last inequality follows because N' is ℓ-cohesiveness and thus $| \cup_{i \in T} A_i| \geq | \cap_{i \in T} A_i| \geq \ell$ for all $T \subseteq N'$. □

PROOF OF PROPOSITION 3.8

Proposition 3.8. *Fractional core implies group resource proportionality.*

Proof. We prove the contrapositive. Let $\vec{p}$ be a fractional committee which does not satisfy GRP and let $S \subseteq N$ be a group of voters for which GRP is violated. We will show the existence of a blocking deviation $\vec{q}$ for the group of voters S and thus show that $\vec{p}$ does not lie in the core.

If $| \bigcup_{i \in S} A_i| \leq |S|\frac{k}{n}$, then let $\vec{q} = \vec{1}_{\cup_{i \in S} A_i}$. It is clear that $\sum_{c \in C} q_c \leq |S|\frac{k}{n}$ and that $u_i(\vec{p}) \leq u_i(\vec{q})$ for all $i \in S$. Since S violates GRP, we have

$$\sum_{c \in \bigcup\limits_{i \in S} A_i} p_c < |S|\frac{k}{n} - \max_{T \subseteq S} \left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right] \leq |\bigcup_{i \in S} A_i|$$

and thus there must be some $i \in S$ for which $u_i(\vec{p}) < u_i(\vec{q})$.

Suppose instead that $| \bigcup_{i \in S} A_i| > |S|\frac{k}{n}$. Note that

$$\sum_{c \in \bigcup\limits_{i \in S} A_i} p_c < |S|\frac{k}{n} - \max_{T \subseteq S} \left[|T|\frac{k}{n} - |\bigcup_{i \in T} A_i|\right] \leq |S|\frac{k}{n} < |\bigcup_{i \in S} A_i|$$

and thus there must be some candidate $c' \in \bigcup_{i \in S} A_i$ such that $p_{c'} < 1$. Let $\epsilon = \min(|S|\frac{k}{n} - \sum_{c \in \bigcup\limits_{i \in S} A_i} p_c, 1 - p_{c'})$. Set $q_{c'} = p_{c'} + \epsilon$, $q_c = p_c$ for all $c \in \bigcup_{i \in S} A_i \setminus \{c'\}$, and $q_c = 0$ otherwise. By construction, $\sum_{c \in C} q_c \leq |S|\frac{k}{n}$. Furthermore, it is clear that $u_i(\vec{p}) \leq u_i(\vec{q})$ for all $i \in S$ and the voters in S who approve of c' will strictly improve. □

PROOF OF PROPOSITION 3.9

Proposition 3.9. *For a given fractional committee $\vec{p}$, checking whether $\vec{p}$ satisfies group resource proportionality can be done in polynomial time.*

Proof. Given fractional committee $\vec{p}$, one can check whether $\vec{p}$ satisfies GRP by the following simple procedure. Let $\mathcal{N}$ be the network formulation of the instance. Modify $\mathcal{N}$ so the capacities on arcs from the candidate set to the sink are $cap(c \rightarrow t) = p_c$ for all $c \in C$, and denote the resulting network $\mathcal{N}'$. Compute a max flow f' on $\mathcal{N}'$ and max flow f on $\mathcal{N}$. Note that, because of the capacity constraints in $\mathcal{N}'$, the fractional committee given by f' is elementwise dominated by $\vec{p}$. Finally, since the flow values of f and f' will be equal if and only if f' is a max flow on $\mathcal{N}$, it follows from Theorem 3.3 that $\vec{p}$ is GRP if and only if the flow values of f and f' are equal. □

PROOF OF LEMMA 4.6

Lemma 4.6. *Let $\{a_i\}$ and $\{b_i\}$ be two sequences of r real numbers such that $a_1 \geq a_2 \geq \ldots \geq a_r \geq 0$ and for all $r' \in [r]$, it holds that $\sum_{i \in [r']} b_i \geq 0$. Then, it must be that $\sum_{i \in [r]} a_i \cdot b_i \geq 0$.*

Proof. Note that for each $j \in [r]$, we can write $b_j = \sum_{i \in [j]} b_i - \sum_{i \in [j-1]} b_i$. Substituting, we can then transform our expression of interest as follows:

$$\begin{aligned}
&\sum_{i \in [r]} a_i \cdot b_i \\
&= a_1 \sum_{i=1}^{1} b_i + (a_2 \sum_{i=1}^{2} b_i - a_2 \sum_{i=1}^{1} b_i) + \ldots + (a_{r-1} \sum_{i=1}^{r-1} b_i - a_{r-1} \sum_{i=1}^{r-2} b_i) + (a_r \sum_{i=1}^{r} b_i - a_r \sum_{i=1}^{r-1} b_i) \\
&\geq a_2 \sum_{i=1}^{1} b_i + (a_3 \sum_{i=1}^{2} b_i - a_2 \sum_{i=1}^{1} b_i) + \ldots + (a_r \sum_{i=1}^{r-1} b_i - a_{r-1} \sum_{i=1}^{r-2} b_i) + (a_r \sum_{i=1}^{r} b_i - a_r \sum_{i=1}^{r-1} b_i) \\
&\geq a_r \sum_{i=1}^{r} b_i \geq 0.
\end{aligned}$$

The first inequality uses the fact that each of the partial sums of $\{b_i\}$ are non-negative and the fact that $\{a_i\}$ are non-increasing to replace $a_j \cdot \sum_{i=1}^{j} b_i$ with the weakly smaller quantity $a_{j+1} \cdot \sum_{i=1}^{j} b_i$ for all $j \in [r-1]$. The second inequality is due to cancellation. The third inequality again follows from the non-negativity of $\{a_i\}$ and of $\sum_{i \in [r]} b_i$. □